Math 564: Real analysis and measure theory
Lecture 9

99% lemma. Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and let $\mathcal{C} \subseteq \mathcal{B}$ be a collection of sets whose finite disjoint unions form an algebra generating $\mathcal{B}$. Then each positive measure set $M \subseteq X$ admits a set $C \in \mathcal{C}$ whose 99% is M, i.e. $\forall \varepsilon > 0$ $\exists C \in \mathcal{C}$ s.t. $\frac{\mu(M \cap C)}{\mu(C)} \geq 1 - \varepsilon$ (think 0.99).

Proof. By the uniqueness part of Carathéodory's theorem $\mu = (\mu|_{\langle \mathcal{C} \rangle})^*$. Thus, $\mu(M) = \inf \{ \sum_{n \in \mathbb{N}} \mu(A_n) : \bigcup_n A_n \supseteq M \text{ and } \{A_n\} \subseteq \langle \mathcal{C} \rangle \}$. By disjointification we may assume the A_n are pairwise disjoint; furthermore, since each A_n is a finite disjoint union of sets in $\mathcal{C}$, we get

$$\mu(M) = \inf \{ \sum_{k \in \mathbb{N}} C_k : \bigsqcup_{k \in \mathbb{N}} C_k \supseteq M \text{ and } \{C_k\} \subseteq \mathcal{C} \}.$$

Using σ-finiteness, M has a μ-meas. subset of positive finite measure, so by shrinking M, we may assume that $\mu(M) < \infty$. Then $\exists \{C_k\} \subseteq \mathcal{C}$ such that $\bigsqcup_{k \in \mathbb{N}} C_k \supseteq M$ and $\frac{\mu(M)}{\mu(\bigsqcup_{k \in \mathbb{N}} C_k)} \geq 1 - \varepsilon$. By the carrot and soup observation, we have $\frac{\mu(M \cap C_k)}{\mu(C_k)} \geq 1 - \varepsilon$ for some $k \in \mathbb{N}$. $\square$

Examples. (a) For $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$, we take $\mathcal{C} :=$ boxes, hence we get that every positive measure set contains 99% of a box C.

(b) For $(A^{\mathbb{N}}, \mathcal{B}(A^{\mathbb{N}}), \mu)$, where A is finite and μ is Bernoulli, we take $\mathcal{C} :=$ cylinders, so every positive measure set contains 99% of a cylinder C.

Note. In both of these examples, we can take the box/cylinder C to be arbitrarily small (both small diameter and small measure because because each box/cylinder partitions into arbitrarily small (finitely many) boxes/cylinders, hence carrot-soup observation applies.

Application: ergodicity

Def. Let (X, μ) be a measure space and let E be an equivalence relation on X. The relation E is called *ergodic* (wrt μ) or *μ-ergodic* if every E-invariant (i.e. union of E-classes) μ-measurable set is null or conull. In other words, X is not decomposible into two E-invariant positive measure sets.

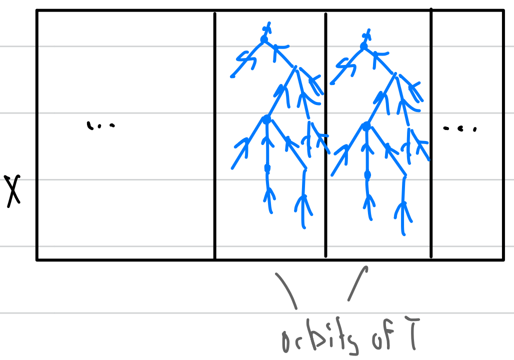
Examples of equiv. rel.

(a) Let Γ be a ctbl group acting on a measure space $(X, \mathcal{B}, \mu)$ so that $\gamma \cdot B \in \mathcal{B}$ for all $\gamma \in \Gamma$ and $B \in \mathcal{B}$. For example translation actions $\mathbb{Z} \curvearrowright \mathbb{R}$ or $\mathbb{Q} \curvearrowright \mathbb{R}$ or dialations $(\mathbb{Q}_{>0}, \cdot) \curvearrowright \mathbb{R}^d$. Then the orbit equiv. rel. on X of this action, denoted E_Γ and defined by

$$x \, E_\Gamma \, y :\Leftrightarrow x \text{ and } y \text{ are in the same } \Gamma\text{-orbit}$$
$$:\Leftrightarrow y = \gamma \cdot x \text{ for some } \gamma \in \Gamma.$$

(b) Let $(X, \mathcal{B}, \mu)$ be a measure space and $T : X \to X$ not necessarily a bijection. Typically we will assume that T is "μ-measurable". Its orbit eq. rel., denoted E_T, is defined by:

$$x \, E_T \, y :\Leftrightarrow T^n x = T^m y \text{ for some } n, m \in \mathbb{N}.$$



We draw an edge $\underset{x}{\bullet} \longrightarrow \underset{T(x)}{\bullet}$. Then the T-orbits are exactly the connected components of this graph, which is the graph of T as a subset of $X \times X$.

Examples of ergodic/nonergodic eq. rel.

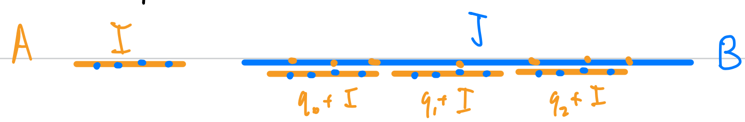
(a) Nonergodic. Let $\mathbb{Z} \curvearrowright \mathbb{R}$ by translation: $z \cdot r := z + r$, for $z \in \mathbb{Z}$, $r \in \mathbb{R}$. Then the orbit eq. rel. is just the coset eq. rel. of $\mathbb{Z} \leq \mathbb{R}$. The orbit of $x \in \mathbb{R}$ is $x + \mathbb{Z}$.

Then $A := (0,\frac{1}{2}) + \mathbb{Z} = \bigsqcup_{n\in\mathbb{Z}} (n, n+\frac{1}{2})$ is $E_{\mathbb{Z}}$-invariant but it and its complement have positive measure so $E_{\mathbb{Z}}$ is not λ-ergodic, where λ is Lebesgue meas. Note that $E_{\mathbb{Z}}$ admits a measurable transversal, e.g. $[0,1)$.

(b) Ergodic. Let $\mathbb{Q} \curvearrowright \mathbb{R}$ by translation, so its orbits eq. rel. $E_{\mathbb{Q}}$ is the coset eq. rel. of $\mathbb{Q} \le \mathbb{R}$. Recall that $E_{\mathbb{Q}}$ doesn't admit a measurable transversal, and the reason for this is that $E_{\mathbb{Q}}$ is ergodic, which we'll prove using the 99% lemma.

Claim. $E_{\mathbb{Q}}$ is ergodic.

Proof. Suppose otherwise, so there is a positive measure $A \subseteq \mathbb{R}$ with $B := \mathbb{R} \setminus A$ of positive measure. By the 99% lemma, there is a positive measure interval J whose 99% is B. By the 99% lemma again, there is a positive measure interval I whose 99% is A and moreover, $\mathrm{lh}(I) < \mathrm{lh}(J)$.



Using that rationals are dense, we can cover $\geq$ half of J by finitely many pairwise disjoint rational translates of I, i.e.

$$\bigsqcup_{i<k} (q_i + I) \subseteq J \quad \text{and} \quad \lambda\Big(\bigsqcup_{i<k} q_i + I\Big) \geq \frac{1}{2}\lambda(J).$$

Since $q_i + A = A$ for all i, we have that 99% of each $q_i + I$ is still A. So $\geq 0.5 \cdot 99\%$ of J is A, contradicting that only $\leq 1\%$ of J is A. $\square$

Regularity of measures (approximating with open/closed sets).

Def. Let $(X, \mathcal{B}, \mu)$ be a measure space and X a metric space. Then μ is called regular if each μ-meas. set M satisfies:

$$\mu(M) = \inf\{\mu(U) : U \supseteq M \text{ open}\} \quad \text{(outer regularity)}$$
$$= \sup\{\mu(C) : C \subseteq M \text{ closed}\}. \quad \text{(inner regularity)}$$

μ is called strongly regular if

$$0 = \inf\{\mu(U \setminus M) : U \supseteq M \text{ open}\}$$
$$= \inf\{\mu(M \setminus C) : C \subseteq M \text{ closed}\}.$$

Obs. All finite regular measures are strongly regular.

Prop. If μ is strongly regular, then every measurable set M is $=_\mu$ G_δ and $=_\mu$ F_σ; more precisely, there are a G_δ set G and an F_σ set F such that
$$F \subseteq M \subseteq G \quad \text{and} \quad F =_\mu M =_\mu G.$$

Proof. By strong regularity, for each $n \in \mathbb{N}^+$ there are an open set U_n and a closed set C_n such that $C_n \subseteq M \subseteq U_n$ and $\mu(M \setminus C_n)$, $\mu(U_n \setminus M) \leq \frac{1}{n}$. Let $G := \bigcap_{n \in \mathbb{N}} U_n$ and $F := \bigcup_{n \in \mathbb{N}} C_n$, so $F \subseteq M \subseteq G$ and $\mu(M \setminus F) \leq \mu(M \setminus C_n) \leq \frac{1}{n} \to 0$ and $\mu(G \setminus M) \leq \mu(U_n \setminus M) \leq \frac{1}{n} \to 0$ as $n \to \infty$, so $F =_\mu M =_\mu G$. $\square$